① (a) $\boxed{\text{True}}$

$15-(-5)=20$ and $4 \mid 20$.

(b) $\boxed{\text{False}}$

$100-4=96$ and $5 \nmid 96$.

(c)

$$\overline{-2}\cdot\overline{5}+\overline{2}^{4}+\overline{2}\cdot\overline{4}\cdot\overline{10} = \overline{-10}+\overline{16}+\overline{80}$$
$$= \overline{6}+\overline{80}$$
$$= \overline{0}+\overline{2} = \boxed{\overline{2}}$$

$$\begin{array}{r} 13 \\ 6\,\overline{)\,80} \\ -6 \\ \hline 20 \\ -18 \\ \hline 2 \end{array}$$

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② (a) $\boxed{\mathbb{Z}_{10}^{\times} = \{\overline{1},\overline{3},\overline{7},\overline{9}\}}$

(b) $\varphi(10)=4$

So, $\overline{3}^{4}=\overline{1}$.

$1002 = 4\cdot 250+2$

So,

$$\overline{3}^{1002} = \left(\overline{3}^{4}\right)^{250}\cdot\overline{3}^{2} = \overline{1}^{250}\cdot\overline{3}^{2} = \boxed{\overline{9}}$$

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③ (a) $\overline{3}\cdot\overline{5}=\overline{15}=\overline{1}$

$\boxed{\overline{3}^{-1}=\overline{5}}$

(b)

$$\left.\begin{aligned}
\overline{3}^{1} &= \overline{3} \\
\overline{3}^{2} &= \overline{9} = \overline{2} \\
\overline{3}^{3} &= \overline{3}\cdot\overline{2} = \overline{6} \\
\overline{3}^{4} &= \overline{3}\cdot\overline{6} = \overline{18} = \overline{4} \\
\overline{3}^{5} &= \overline{3}\cdot\overline{4} = \overline{12} = \overline{5} \\
\overline{3}^{6} &= \overline{3}\cdot\overline{5} = \overline{15} = \overline{1}
\end{aligned}\right\}$$

$\boxed{\text{So, } \overline{3} \text{ is a primitive root}}$

(A) HW 3 – #1(c)

(B) HW 5 – #11

(C) Suppose $x^2 + x \equiv 2 \pmod{p}$ where p is an odd prime.

Then, $p \mid (x^2 + x - 2)$.

So, $p \mid (x+2)(x-1)$

Since p is prime we get $p \mid (x+2)$ or $p \mid (x-1)$.

So, $x \equiv -2 \pmod{p}$ or $x \equiv 1 \pmod{p}$.

(D) Suppose there exist integers x and y where $15x^2 - 10x + 7y^2 = 11$.

Then in $\mathbb{Z}_5$ we would have $\overline{15}\,\bar{x}^2 - \overline{10}\,x + \bar{7}\,\bar{y}^2 = \overline{11}$

So, $\bar{0}\,\bar{x}^2 + \bar{0}\,\bar{x} + \bar{2}\,\bar{y}^2 = \bar{1}$ in $\mathbb{Z}_5$.

So, $\bar{2}\,\bar{y}^2 = \bar{1}$ in $\mathbb{Z}_5$.

There is no such $\bar{y}$ is $\mathbb{Z}_5$ by this table

$\bar{y}$	$\bar{2}\,\bar{y}^2$
$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{8} = \bar{3}$
$\bar{3}$	$\overline{18} = \bar{3}$
$\bar{4}$	$\overline{32} = \bar{2}$

never get $\bar{1}$

Thus, there are no $x, y \in \mathbb{Z}$ with $15x^2 - 10x + 7y^2 = 11$.